Simplification Techniques

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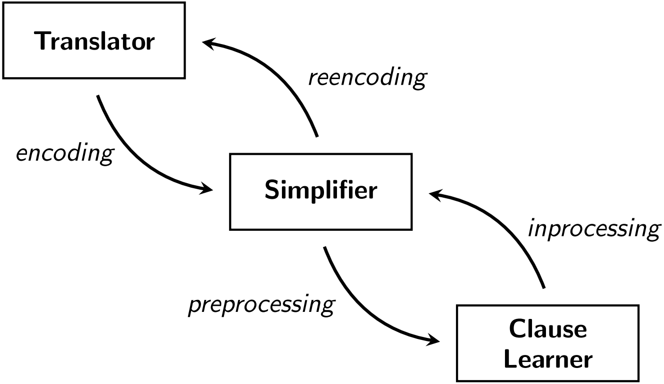
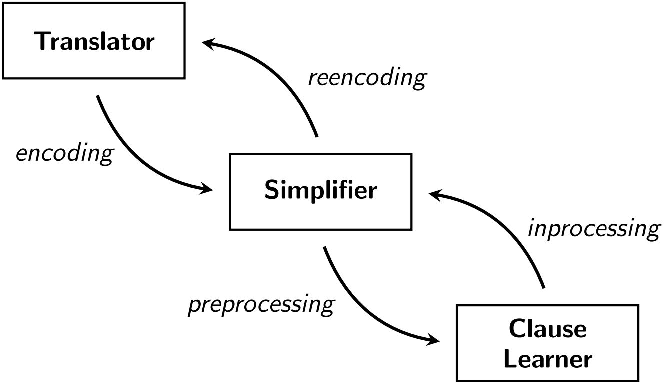
Interaction between different solving approaches



Interaction between different solving approaches



It all comes down to adding and removing redundant clauses

Redundant clauses

A clause is redundant with respect to a formula if adding it to the formula preserves satisfiability.

- For unsatisfiable formulas, all clauses can be added, including the empty clause $\bot$.

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$$\frac{C \lor x \qquad \bar{x} \lor D}{C \lor D}$$

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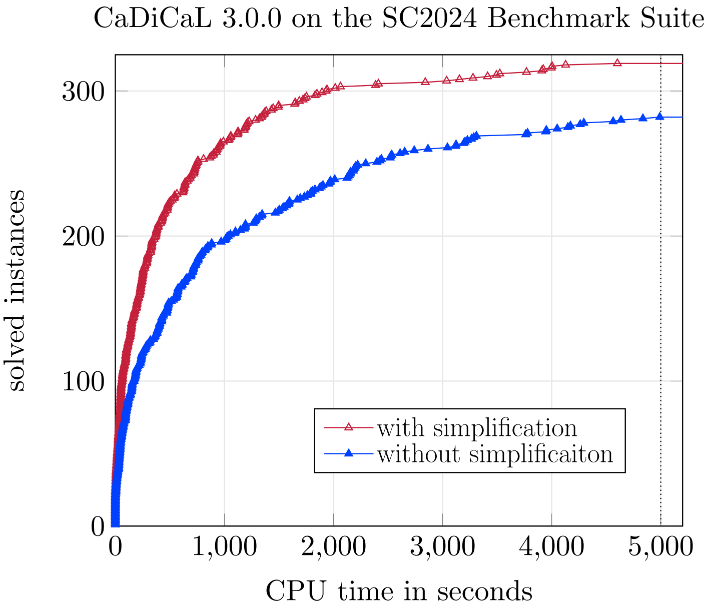
- For satisfiable formulas, all clauses can be removed.

Challenge regarding redundant clauses:

- How to check redundancy in polynomial time?
- Ideally find redundant clauses in linear time

Simplification in Practice



Tautologies and Subsumption

Definition (Tautology)

A clause C is a tautology if its contains two complementary literals x and $\overline{x}$.

Example

The clause $(a \vee b \vee \overline{b})$ *is a tautology.*

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \overline{e})$.

Self-Subsuming Resolution

$$\frac{C \lor x \qquad D \lor \overline{x}}{D}\; C \subseteq D \qquad \frac{(a \lor b \lor x) \qquad (a \lor b \lor e \lor \overline{x})}{(a \lor b \lor e)}$$

resolvent D subsumes second antecedent $D \lor \overline{x}$

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Example

Assume a CNF contains both antecedents

$\ldots (a \lor b \lor x)(a \lor b \lor e \lor \bar{x}) \ldots$

If D *is added, then* $D \lor \bar{x}$ *can be removed*

which in essence removes $\bar{x}$ *from* $D \lor \bar{x}$

$\ldots (a \lor b \lor x)(a \lor b \lor e \quad) \ldots$

Initially in the SATeLite preprocessor, [EenBiere'07]
now common in most solvers (i.e., as pre- and inprocessing)

Self-Subsuming Example

Self-Subsuming Resolution

$$\frac{C \vee x \qquad D \vee \overline{x}}{D}\; C \subseteq D \qquad \frac{(a \vee b \vee x) \qquad (a \vee b \vee e \vee \overline{x})}{(a \vee b \vee e)}$$

resolvent D subsumes second antecedent $D \vee \overline{x}$

Example: Remove literals using self-subsumption

$$\begin{array}{lll}
(a \vee b \vee e) & \wedge & (\overline{a} \vee b \vee e) \wedge \\
(\overline{a} \vee b \vee \overline{e}) & \wedge & (a \vee \overline{b} \vee e) \wedge \\
(\overline{a} \vee \overline{b} \vee f) & \wedge & (\overline{a} \vee \overline{b} \vee \overline{f}) \wedge \\
(a \vee \overline{e} \vee f) & \wedge & (a \vee \overline{e} \vee \overline{f})
\end{array}$$

Self-Subsuming Example

Self-Subsuming Resolution

$$\frac{C \vee x \qquad D \vee \overline{x}}{D} \; C \subseteq D \qquad \frac{(a \vee b \vee x) \qquad (a \vee b \vee e \vee \overline{x})}{(a \vee b \vee e)}$$

resolvent D subsumes second antecedent $D \vee \overline{x}$

Example: Remove literals using self-subsumption

$$\begin{array}{l} (\quad b \vee e) \wedge (\overline{a} \vee b \vee e) \wedge \\ (\overline{a} \vee b \vee \overline{e}) \wedge (a \vee \overline{b} \vee e) \wedge \\ (\overline{a} \vee \overline{b} \vee f) \wedge (\overline{a} \vee \overline{b} \vee \overline{f}) \wedge \\ (a \vee \overline{e} \vee f) \wedge (a \vee \overline{e} \vee \overline{f}) \end{array}$$

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\end{array}$$

Implementing Subsumption

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \lor b)$ *subsumes clause* $(a \lor b \lor \overline{e})$*.*

Forward Subsumption

for each clause C in formula Γ **do**
  if C is subsumed by a clause D in $\Gamma \setminus C$ **then**
    remove C from Γ

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Backward Subsumption

for each clause C in formula Γ **do**
  remove all clauses D in Γ that are subsumed by C

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Backward Subsumption

for each clause C in formula Γ **do**
  pick a literal x in C
  remove all clauses D in Γ_x that are subsumed by C

Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \vee a_1 \vee \dots \vee a_i)$ and $D = (\overline{x} \vee b_1 \vee \dots \vee b_j)$, the *resolvent* of C and D on variable x (denoted by $C \bowtie_x D$) is $(a_1 \vee \dots \vee a_i \vee b_1 \vee \dots \vee b_j)$

Resolution on sets of clauses Γ_x and $\Gamma_{\overline{x}}$ (denoted by $\Gamma_x \bowtie_x \Gamma_{\overline{x}}$) generates all non-tautological resolvents of $C \in \Gamma_x$ and $D \in \Gamma_{\overline{x}}$.

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Proof procedure [DavisPutnam'60]

VE is a complete proof procedure. Applying VE until fixpoint results in either the empty formula (satisfiable) or empty clause (unsatisfiable)

Example VE by clause distribution [DavisPutnam'60]

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Example of clause distribution

$$
\begin{array}{rc|ccc}
 & & \multicolumn{3}{c}{\overbrace{(x \vee c) \qquad (x \vee \overline{d}) \qquad (x \vee \overline{a} \vee \overline{b})}^{\Gamma_x}} \\
\hline
\multirow{3}{*}{$\Gamma_{\overline{x}}$} & (\overline{x} \vee a) & (a \vee c) & (a \vee \overline{d}) & (a \vee \overline{a} \vee \overline{b}) \\
 & (\overline{x} \vee b) & (b \vee c) & (b \vee \overline{d}) & (b \vee \overline{a} \vee \overline{b}) \\
 & (\overline{x} \vee \overline{e} \vee f) & (c \vee \overline{e} \vee f) & (\overline{d} \vee \overline{e} \vee f) & (\overline{a} \vee \overline{b} \vee \overline{e} \vee f)
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		Γ_x		
		$(x \vee c)$	$(x \vee \overline{d})$	$(x \vee \overline{a} \vee \overline{b})$
$\Gamma_{\overline{x}}$	$(\overline{x} \vee a)$	$(a \vee c)$	$(a \vee \overline{d})$	$\cancel{(a \vee \overline{a} \vee \overline{b})}$
	$(\overline{x} \vee b)$	$(b \vee c)$	$(b \vee \overline{d})$	$\cancel{(b \vee \overline{a} \vee \overline{b})}$
	$(\overline{x} \vee \overline{e} \vee f)$	$(c \vee \overline{e} \vee f)$	$(\overline{d} \vee \overline{e} \vee f)$	$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

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Example of clause distribution

	Γ_x		
	$(x \lor c)$	$(x \lor \overline{d})$	$(x \lor \overline{a} \lor \overline{b})$
$\Gamma_{\overline{x}}$: $(\overline{x} \lor a)$	$(a \lor c)$	$(a \lor \overline{d})$	$\cancel{(a \lor \overline{a} \lor \overline{b})}$
$\Gamma_{\overline{x}}$: $(\overline{x} \lor b)$	$(b \lor c)$	$(b \lor \overline{d})$	$\cancel{(b \lor \overline{a} \lor \overline{b})}$
$\Gamma_{\overline{x}}$: $(\overline{x} \lor \overline{e} \lor f)$	$(c \lor \overline{e} \lor f)$	$(\overline{d} \lor \overline{e} \lor f)$	$(\overline{a} \lor \overline{b} \lor \overline{e} \lor f)$

In the example: $|\Gamma_x \bowtie \Gamma_{\overline{x}}| > |\Gamma_x| + |\Gamma_{\overline{x}}|$

Exponential growth of clauses in general

VE by substitution [EenBiere'07]

General idea

Detect gates (or definitions) $x = \mathrm{GATE}(a_1, \ldots, a_n)$ in the formula and use them to reduce the number of added clauses

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Possible gates

gate	G_x	$G_{\overline{x}}$
$\mathrm{AND}(a_1, \ldots, a_n)$	$(x \vee \overline{a}_1 \vee \cdots \vee \overline{a}_n)$	$(\overline{x} \vee a_1), \ldots, (\overline{x} \vee a_n)$
$\mathrm{OR}(a_1, \ldots, a_n)$	$(x \vee \overline{a}_1), \ldots, (x \vee \overline{a}_n)$	$(\overline{x} \vee a_1 \vee \cdots \vee a_n)$
$\mathrm{ITE}(c, t, f)$	$(x \vee \overline{c} \vee \overline{t}), (x \vee c \vee \overline{f})$	$(\overline{x} \vee \overline{c} \vee t), (\overline{x} \vee c \vee f)$

VE by substitution [EenBiere'07]

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$\mathrm{OR}(a_1, \dots, a_n)$	$(x \vee \overline{a}_1), \dots, (x \vee \overline{a}_n)$	$(\overline{x} \vee a_1 \vee \dots \vee a_n)$
$\mathrm{ITE}(c, t, f)$	$(x \vee \overline{c} \vee \overline{t}), (x \vee c \vee \overline{f})$	$(\overline{x} \vee \overline{c} \vee t), (\overline{x} \vee c \vee f)$

Variable elimination by substitution [EenBiere'07]

Let $R_x = \Gamma_x \setminus G_x$; $R_{\overline{x}} = \Gamma_{\overline{x}} \setminus G_{\overline{x}}$.

Replace $\Gamma_x \wedge \Gamma_{\overline{x}}$ by $G_x \bowtie_x R_{\overline{x}} \wedge G_{\overline{x}} \bowtie_x R_x$.
Always less than $\Gamma_x \bowtie_x \Gamma_{\overline{x}}$!

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \mathrm{AND}(a, b)$

$$\Gamma_x = (x \vee c) \wedge (x \vee \overline{d}) \wedge (x \vee \overline{a} \vee \overline{b})$$
$$\Gamma_{\overline{x}} = (\overline{x} \vee a) \wedge (\overline{x} \vee b) \wedge (\overline{x} \vee \overline{e} \vee f)$$

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Example of substitution

$$\begin{array}{rc|ccc}
 & & \multicolumn{2}{c}{\overbrace{\hspace{12em}}^{R_x}} & \overbrace{\hspace{6em}}^{G_x} \\
 & & (x \vee c) & (x \vee \overline{d}) & (x \vee \overline{a} \vee \overline{b}) \\
\hline
G_{\overline{x}} \left\{ \right. & (\overline{x} \vee a) & (a \vee c) & (a \vee \overline{d}) & \\
 & (\overline{x} \vee b) & (b \vee c) & (b \vee \overline{d}) & \\
R_{\overline{x}} \left\{ \right. & (\overline{x} \vee \overline{e} \vee f) & & & (\overline{a} \vee \overline{b} \vee \overline{e} \vee f)
\end{array}$$

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Example of substitution

		R_x: $(x \vee c)$	R_x: $(x \vee \overline{d})$	G_x: $(x \vee \overline{a} \vee \overline{b})$
$G_{\overline{x}}$	$(\overline{x} \vee a)$	$(a \vee c)$	$(a \vee \overline{d})$	
$G_{\overline{x}}$	$(\overline{x} \vee b)$	$(b \vee c)$	$(b \vee \overline{d})$	
$R_{\overline{x}}$	$(\overline{x} \vee \overline{e} \vee f)$			$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

using substitution: $|\Gamma_x \bowtie \Gamma_{\overline{x}}| < |\Gamma_x| + |\Gamma_{\overline{x}}|$

Bounded Variable Addition

Main Idea

Given a CNF formula Γ, can we construct a (semi)logically equivalent Γ' by introducing a new variable $x \notin VAR(\Gamma)$ such that $|\Gamma'| < |\Gamma|$?

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Reverse of Variable Elimination

For example, replace the clauses

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\begin{array}{ccc}
(a \vee c) & (a \vee \overline{d}) & \\
(b \vee c) & (b \vee \overline{d}) & \\
(c \vee \overline{e} \vee f) & (\overline{d} \vee \overline{e} \vee f) & (\overline{a} \vee \overline{b} \vee \overline{e} \vee f)
\end{array}
$$

by

$$
\begin{array}{ccc}
(\overline{x} \vee a) & (\overline{x} \vee b) & (\overline{x} \vee \overline{e} \vee f) \\
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\begin{array}{ccc}
(\overline{x} \lor a) & (\overline{x} \lor b) & (\overline{x} \lor \overline{e} \lor f) \\
(x \lor c) & (x \lor \overline{d}) & (x \lor \overline{a} \lor \overline{b})
\end{array}
$$

Challenge: how to find suitable patterns for replacement?

Factoring Out Subclauses

Example

Replace

$$(a \lor b \lor c \lor d) \quad (a \lor b \lor c \lor e) \quad (a \lor b \lor c \lor f)$$

by

$$(x \lor d) \quad (x \lor e) \quad (x \lor f) \quad (\bar{x} \lor a \lor b \lor c)$$

adds 1 variable and 1 clause *reduces* number of literals by 2

Not compatible with VE, which would eliminate x immediately!

… *so this does not work* …

Bounded Variable Addition

Example

Smallest pattern that is compatible: Replace

$$\begin{array}{cc} (a \lor d) & (a \lor e) \\ (b \lor d) & (b \lor e) \\ (c \lor d) & (c \lor e) \end{array}$$

by

$$\begin{array}{ccc} (\overline{x} \lor a) & (\overline{x} \lor b) & (\overline{x} \lor c) \\ (x \lor d) & (x \lor e) & \end{array}$$

adds 1 variable

removes 1 clause

Bounded Variable Addition

Possible Patterns

$$\begin{matrix} (X_1 \lor L_1) & \dots & (X_1 \lor L_k) \\ \vdots & & \vdots \\ (X_n \lor L_1) & \dots & (X_n \lor L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \lor L_j)$$

replaced by $\bigwedge_{i=1}^{n} (y \lor X_i) \land \bigwedge_{j=1}^{k} (\overline{y} \lor L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j

Bounded Variable Addition

Possible Patterns

$$\begin{matrix} (X_1 \lor L_1) & \dots & (X_1 \lor L_k) \\ \vdots & & \vdots \\ (X_n \lor L_1) & \dots & (X_n \lor L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \lor L_j)$$

replaced by $$\bigwedge_{i=1}^{n} (y \lor X_i) \;\land\; \bigwedge_{j=1}^{k} (\overline{y} \lor L_j)$$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j
- Reduction: $nk - n - k$ clauses are removed by replacement

Bounded Variable Addition on AtMostOne (1)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee \overline{x}_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_6) \wedge (\overline{x}_2 \vee \overline{x}_6) \wedge (\overline{x}_3 \vee \overline{x}_6) \wedge (\overline{x}_4 \vee \overline{x}_6) \wedge (\overline{x}_5 \vee \overline{x}_6) \wedge \\
(\overline{x}_1 \vee \overline{x}_7) \wedge (\overline{x}_2 \vee \overline{x}_7) \wedge (\overline{x}_3 \vee \overline{x}_7) \wedge (\overline{x}_4 \vee \overline{x}_7) \wedge (\overline{x}_5 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_8) \wedge (\overline{x}_2 \vee \overline{x}_8) \wedge (\overline{x}_3 \vee \overline{x}_8) \wedge (\overline{x}_4 \vee \overline{x}_8) \wedge (\overline{x}_5 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_9) \wedge (\overline{x}_2 \vee \overline{x}_9) \wedge (\overline{x}_3 \vee \overline{x}_9) \wedge (\overline{x}_4 \vee \overline{x}_9) \wedge (\overline{x}_5 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_{10}) \wedge (\overline{x}_2 \vee \overline{x}_{10}) \wedge (\overline{x}_3 \vee \overline{x}_{10}) \wedge (\overline{x}_4 \vee \overline{x}_{10}) \wedge (\overline{x}_5 \vee \overline{x}_{10})
\end{array}
$$

Bounded Variable Addition on AtMostOne (1)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee \overline{x}_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
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(\overline{x}_1 \vee \overline{x}_7) \wedge (\overline{x}_2 \vee \overline{x}_7) \wedge (\overline{x}_3 \vee \overline{x}_7) \wedge (\overline{x}_4 \vee \overline{x}_7) \wedge (\overline{x}_5 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_8) \wedge (\overline{x}_2 \vee \overline{x}_8) \wedge (\overline{x}_3 \vee \overline{x}_8) \wedge (\overline{x}_4 \vee \overline{x}_8) \wedge (\overline{x}_5 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_9) \wedge (\overline{x}_2 \vee \overline{x}_9) \wedge (\overline{x}_3 \vee \overline{x}_9) \wedge (\overline{x}_4 \vee \overline{x}_9) \wedge (\overline{x}_5 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_{10}) \wedge (\overline{x}_2 \vee \overline{x}_{10}) \wedge (\overline{x}_3 \vee \overline{x}_{10}) \wedge (\overline{x}_4 \vee \overline{x}_{10}) \wedge (\overline{x}_5 \vee \overline{x}_{10})
\end{array}
$$

Replace $(\overline{x}_i \vee \overline{x}_j)$ with $i \in \{1..5\}, j \in \{6..10\}$ by $(\overline{x}_i \vee y)$, $(\overline{x}_j \vee \overline{y})$

Bounded Variable Addition on AtMostOne (2)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \lor \overline{x}_2) \land (\overline{x}_9 \lor \overline{x}_{10}) \land (\overline{x}_8 \lor \overline{x}_{10}) \land (\overline{x}_7 \lor \overline{x}_{10}) \land (\overline{x}_6 \lor \overline{x}_{10}) \land \\
(\overline{x}_1 \lor \overline{x}_3) \land (\overline{x}_2 \lor \overline{x}_3) \land (\overline{x}_8 \lor \overline{x}_9) \land (\overline{x}_7 \lor \overline{x}_9) \land (\overline{x}_6 \lor \overline{x}_9) \land \\
(\overline{x}_1 \lor \overline{x}_4) \land (\overline{x}_2 \lor \overline{x}_4) \land (\overline{x}_3 \lor \overline{x}_4) \land (\overline{x}_7 \lor \overline{x}_8) \land (\overline{x}_6 \lor \overline{x}_8) \land \\
(\overline{x}_1 \lor \overline{x}_5) \land (\overline{x}_2 \lor \overline{x}_5) \land (\overline{x}_3 \lor \overline{x}_5) \land (\overline{x}_4 \lor \overline{x}_5) \land (\overline{x}_6 \lor \overline{x}_7) \land \\
(\overline{x}_1 \lor y) \land (\overline{x}_2 \lor y) \land (\overline{x}_3 \lor y) \land (\overline{x}_4 \lor y) \land (\overline{x}_5 \lor y) \land \\
(\overline{x}_6 \lor \overline{y}) \land (\overline{x}_7 \lor \overline{y}) \land (\overline{x}_8 \lor \overline{y}) \land (\overline{x}_9 \lor \overline{y}) \land (\overline{x}_{10} \lor \overline{y})
\end{array}
$$

Bounded Variable Addition on AtMostOne (2)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee \overline{x}_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee y) \wedge (\overline{x}_2 \vee y) \wedge (\overline{x}_3 \vee y) \wedge (\overline{x}_4 \vee y) \wedge (\overline{x}_5 \vee y) \wedge \\
(\overline{x}_6 \vee \overline{y}) \wedge (\overline{x}_7 \vee \overline{y}) \wedge (\overline{x}_8 \vee \overline{y}) \wedge (\overline{x}_9 \vee \overline{y}) \wedge (\overline{x}_{10} \vee \overline{y})
\end{array}
$$

Replace matched pattern

$$
\begin{array}{l}
(\overline{x}_1 \vee z) \wedge (\overline{x}_2 \vee z) \wedge (\overline{x}_3 \vee z) \wedge \\
(\overline{x}_4 \vee \overline{z}) \wedge (\overline{x}_5 \vee \overline{z}) \wedge (y \vee \overline{z})
\end{array}
$$

Bounded Variable Addition on AtMostOne (3)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee z) \wedge (\overline{x}_2 \vee z) \wedge (\overline{x}_3 \vee z) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_4 \vee \overline{z}) \wedge (\overline{x}_5 \vee \overline{z}) \wedge (y \vee \overline{z}) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
(\overline{x}_4 \vee y) \wedge (\overline{x}_5 \vee y) \wedge (\overline{x}_6 \vee \overline{y}) \wedge (\overline{x}_7 \vee \overline{y}) \wedge (\overline{x}_8 \vee \overline{y}) \\
(\overline{x}_9 \vee \overline{y}) \wedge (\overline{x}_{10} \vee \overline{y})
\end{array}
$$

Bounded Variable Addition on AtMostOne (3)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$(\overline{x}_1 \lor \overline{x}_2) \land (\overline{x}_9 \lor \overline{x}_{10}) \land (\overline{x}_8 \lor \overline{x}_{10}) \land (\overline{x}_7 \lor \overline{x}_{10}) \land (\overline{x}_6 \lor \overline{x}_{10}) \land$$
$$(\overline{x}_1 \lor \overline{x}_3) \land (\overline{x}_2 \lor \overline{x}_3) \land (\overline{x}_8 \lor \overline{x}_9) \land (\overline{x}_7 \lor \overline{x}_9) \land (\overline{x}_6 \lor \overline{x}_9) \land$$
$$(\overline{x}_1 \lor z) \land (\overline{x}_2 \lor z) \land (\overline{x}_3 \lor z) \land (\overline{x}_7 \lor \overline{x}_8) \land (\overline{x}_6 \lor \overline{x}_8) \land$$
$$(\overline{x}_4 \lor \overline{z}) \land (\overline{x}_5 \lor \overline{z}) \land (y \lor \overline{z}) \land (\overline{x}_4 \lor \overline{x}_5) \land (\overline{x}_6 \lor \overline{x}_7) \land$$
$$(\overline{x}_4 \lor y) \land (\overline{x}_5 \lor y) \land (\overline{x}_6 \lor \overline{y}) \land (\overline{x}_7 \lor \overline{y}) \land (\overline{x}_8 \lor \overline{y})$$
$$(\overline{x}_9 \lor \overline{y}) \land (\overline{x}_{10} \lor \overline{y})$$

Replace matched pattern

$$(\overline{x}_6 \lor w) \land (\overline{x}_7 \lor w) \land (\overline{x}_8 \lor w) \land$$
$$(\overline{x}_9 \lor \overline{w}) \land (\overline{x}_{10} \lor \overline{w}) \land (\overline{y} \lor \overline{w})$$

Blocked Clauses [Kullmann'99]

Definition (Blocked Clause)

A clause $(C \vee x)$ is a blocked on x w.r.t. a CNF formula Γ if for every clause $(D \vee \overline{x}) \in \Gamma$, resolvent $C \vee D$ is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{e}) \wedge (\overline{a} \vee e)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{e}$.
Third clause is blocked by e

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Example

Consider the formula $(a \lor b) \land (a \lor \overline{b} \lor \overline{e}) \land (\overline{a} \lor e)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{e}$.
Third clause is blocked by e

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF Γ, remove C from Γ.

Example

Consider $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{e}) \wedge (\overline{a} \vee e)$.
After removing either $(a \vee \overline{b} \vee \overline{e})$ *or* $(\overline{a} \vee e)$, *the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\overline{b}$ *or* $\overline{a}$*).*

An extreme case in which BCE *removes all clauses!*

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While there is a blocked clause C in a CNF Γ, remove C from Γ.

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Consider $(a \lor b) \land (a \lor \overline{b} \lor \overline{e}) \land (\overline{a} \lor e)$.
After removing either $(a \lor \overline{b} \lor \overline{e})$ *or* $(\overline{a} \lor e)$, *the clause* $(a \lor b)$ *becomes blocked (no clause with either* $\overline{b}$ *or* $\overline{a}$*).*

An extreme case in which BCE *removes all clauses!*

Proposition

BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

- BCE converts the Tseitin encoding to Plaisted Greenbaum
- BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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Example of circuit simplification by BCE on Tseitin encoding

(y)

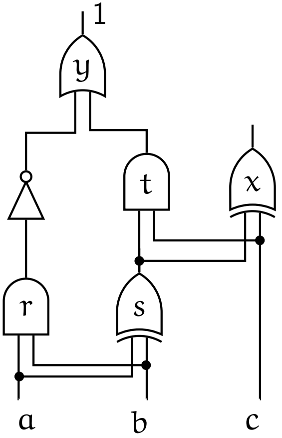
$(\overline{y} \vee t \vee \overline{r})$
$(y \vee \overline{t})$
$(y \vee r)$

$(\overline{x} \vee s \vee c)$
$(\overline{x} \vee \overline{s} \vee \overline{c})$
$(x \vee s \vee \overline{c})$
$(x \vee \overline{s} \vee c)$

$(t \vee \overline{s} \vee \overline{c})$
$(\overline{t} \vee s)$
$(\overline{t} \vee c)$

$(r \vee \overline{a} \vee \overline{b})$
$(\overline{r} \vee a)$
$(\overline{r} \vee b)$

$(\overline{s} \vee a \vee b)$
$(\overline{s} \vee \overline{a} \vee \overline{b})$
$(s \vee a \vee \overline{b})$
$(s \vee \overline{a} \vee b)$



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(y)

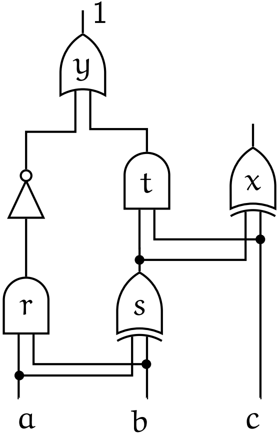
$(\bar{y} \vee t \vee \bar{r})$
$\cancel{(y \vee \bar{t})}$
$\cancel{(y \vee r)}$

$(\bar{x} \vee s \vee c)$
$(\bar{x} \vee \bar{s} \vee \bar{c})$
$(x \vee s \vee \bar{c})$
$(x \vee \bar{s} \vee c)$

$(t \vee \bar{s} \vee \bar{c})$
$(\bar{t} \vee s)$
$(\bar{t} \vee c)$

$(r \vee \bar{a} \vee \bar{b})$
$(\bar{r} \vee a)$
$(\bar{r} \vee b)$

$(\bar{s} \vee a \vee b)$
$(\bar{s} \vee \bar{a} \vee \bar{b})$
$(s \vee a \vee \bar{b})$
$(s \vee \bar{a} \vee b)$



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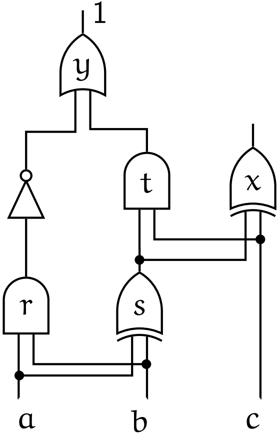
$(\overline{y} \lor t \lor \overline{r})$
$\cancel{(y \lor \overline{t})}$
$\cancel{(y \lor r)}$

$\cancel{(\overline{x} \lor s \lor c)}$
$\cancel{(\overline{x} \lor \overline{s} \lor \overline{c})}$
$\cancel{(x \lor s \lor \overline{c})}$
$\cancel{(x \lor \overline{s} \lor c)}$

$(t \lor \overline{s} \lor \overline{c})$
$(\overline{t} \lor s)$
$(\overline{t} \lor c)$

$(r \lor \overline{a} \lor \overline{b})$
$(\overline{r} \lor a)$
$(\overline{r} \lor b)$

$(\overline{s} \lor a \lor b)$
$(\overline{s} \lor \overline{a} \lor \overline{b})$
$(s \lor a \lor \overline{b})$
$(s \lor \overline{a} \lor b)$



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$\cancel{(x \lor s \lor \bar{c})}$
$\cancel{(x \lor \bar{s} \lor c)}$

$\cancel{(t \lor \bar{s} \lor \bar{c})}$
$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

$(r \lor \bar{a} \lor \bar{b})$
$(\bar{r} \lor a)$
$(\bar{r} \lor b)$

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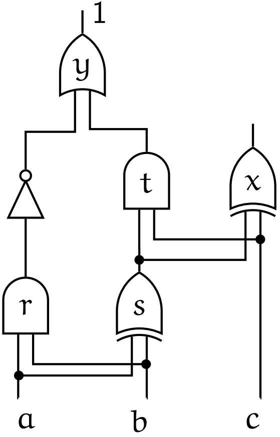

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$\cancel{(y \lor r)}$

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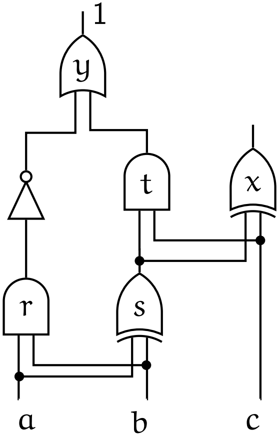
$\cancel{(\bar{x} \lor \bar{s} \lor \bar{c})}$
$\cancel{(x \lor s \lor \bar{c})}$
$\cancel{(x \lor \bar{s} \lor c)}$

$\cancel{(t \lor \bar{s} \lor \bar{c})}$
$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

$(r \lor \bar{a} \lor \bar{b})$
$\cancel{(\bar{r} \lor a)}$
$\cancel{(\bar{r} \lor b)}$

$(\bar{s} \lor a \lor b)$
$(\bar{s} \lor \bar{a} \lor \bar{b})$
$(s \lor a \lor \bar{b})$
$(s \lor \bar{a} \lor b)$



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$\cancel{(x \lor s \lor \bar{c})}$
$\cancel{(x \lor \bar{s} \lor c)}$

$\cancel{(t \lor \bar{s} \lor \bar{c})}$
$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

$(r \lor \bar{a} \lor \bar{b})$
$\cancel{(\bar{r} \lor a)}$
$\cancel{(\bar{r} \lor b)}$

$(\bar{s} \lor a \lor b)$
$(\bar{s} \lor \bar{a} \lor \bar{b})$
$\cancel{(s \lor a \lor \bar{b})}$
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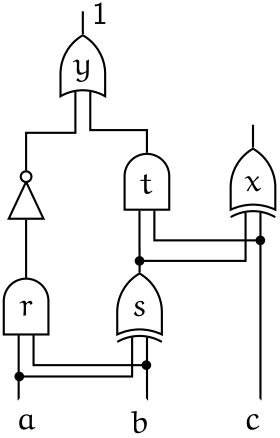

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum

BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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$\cancel{(y \lor r)}$

$\cancel{(\bar{x} \lor s \lor c)}$
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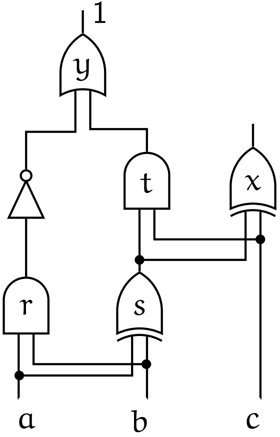

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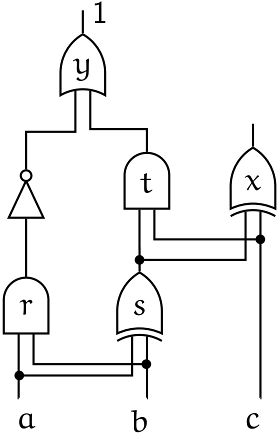

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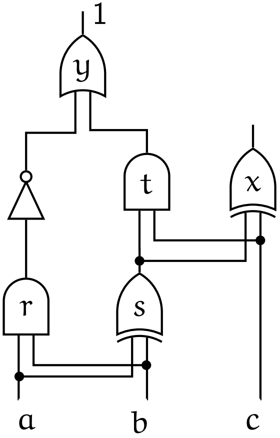

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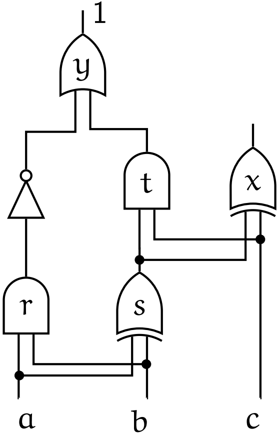

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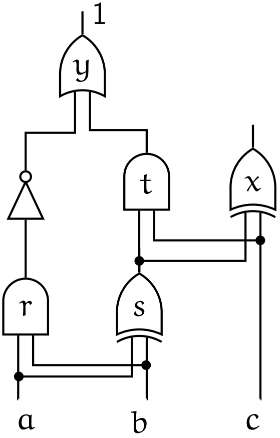

BCE: Solution Reconstruction

Input:

- stack S of eliminated blocked clauses
- formula Γ (without the blocked clauses)
- assignment α that satisfies Γ

Output: an assignment that satisfies $\Gamma \wedge S$

while S.size () **do**
    $\langle C, l \rangle :=$ S.pop ()
    if α falsifies C **then** $\alpha := \alpha_l$
end while
return α

Many Techniques

Many pre- or in-processing techniques in SAT solvers:

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- Variable Elimination
- Blocked Clause Elimination
- Hyper Binary Resolution
- Bounded Variable Addition
- Equivalent Literal Substitution
- Failed Literal Elimination
- Autarky Reasoning
- ...

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... and the list is growing:

- Factoring Learned Clauses [SAT'26]